Exploring Justifications and Enactment of Justification Curriculum in Elementary Classrooms

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Standards for mathematics teaching require teachers to employ teaching practices that promote justification of mathematical ideas. This expected teaching practice is situated in substantial research on students' and teachers' difficulties with justifying mathematical ideas. This study shows different ways elementary school students in grades three through five may justify mathematical conjectures about pattern-finding activities. It also shows that even when teachers are capable of justifying particular tasks, enactment of such tasks in ways that encourage students to go beyond example-based justifications may be problematic. Video and audiotapes of class activities, students' written work, and curriculum materials were sources of data.

Introduction

Evidence that mathematics is a career gatekeeper continues to grow (Berenson, Michaels, Store, 2009; Mendick, 2005; National Council of Teachers of Mathematics (NCTM), 2000) with growing international concerns that many students do not pursue higher mathematics levels (Herzig, 2004; Horn, 2008; Mendick, 2005). Several factors have been claimed to contribute to this problem. These factors include an emphasis in mathematics classrooms on memorization of rules other than mathematical reasoning—a practice that leads to difficulties in learning mathematics (Bergqvist & Lithner, 2012). These problems call for classroom practices that focus on supporting mathematical

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reasoning. "An emphasis on reasoning at all levels of mathematics education calls attention to mathematical argumentation and justification" (Yackel & Hanna, 2003, p. 228).

Justifications can be informal or formal (Harel & Sowder, 2007). Formal justifications are justifications that are typically referred to as mathematical proofs, and reflect the rigor and rules used by expert mathematicians when proving. Informal justifications are relatively less rigorous and can be considered typical in elementary schools whereby students have not been formally instructed in proofs. Conceptions of justification stem from the definitions of proofs that remain controversial because mathematicians and mathematics educators have different views of what constitutes proof and how proofs should be classified (Harel &Sowder, 1998; 2007). In this study, justifications are defined as ways of verifying that a mathematical conjecture is true or false to ascertain for self or persuade others (Bell, 1976; Harel & Sowder, 1998; Healy & Hoyles, 2000). Justifications are subjective and socially constructed as individuals and communities construct their own convictions while they participate in mathematical practices.

Justifications serve several purposes. They support students' understanding of generalizations in algebraic reasoning contexts (Lannin, 2005). Justifications also help in supporting students' sense making (Hiebert, 1997), developing conceptual understanding (Hanna, 2000; Tsamir et al., 2009), and preparing students for proof related higher-level mathematics. Yackel and Hanna (2003) further explained that functions of justifications include "…explanation, systemization, discovery, communication, construction of empirical theory, exploration of definition and of the consequences of assumptions, and incorporation of a well known fact into a framework" (p. 228). It can be argued, then, that learning and justifying are inseparable. To that extent, NCTM (2000), Common Core State Standards for Mathematics (2010), and United Kingdom national standards of education, among other organizations and nations, require students as early as in

elementary schools (before grade or standard seven) to provide justifications for mathematical propositions. Thus, justification is at the core of mathematics education internationally.

Russel, Schifter and Bastable (2011) observed that, despite emphasis on making and justifying general mathematical claims in curriculum standards, justifying proves to be a very difficult process for most students. Students tend to conclude that particular mathematical claims work out for all instances after finding just a few examples that support the claim or they base their arguments on their perceptions (i.e., use empirical justifications). For example, Healy and Hoyles (2000) reported high school students' tendency to use empirical justifications. Ellis's (2007) and Lannin's (2005) studies with middle school students reported the same tendencies for students justifying conjectures from pattern-finding activities. Studies on conceptions of proof aligned with the findings on students' justification tendencies. Knuth (2002) and Kuchemann and Hoyles (2009) studied teachers' and students' conceptions of proof respectively. These studies showed that empirical justifications are favored.

Classroom participants are more able to evaluate, understand, and integrate in their reasoning those justifications that have more explanatory power. Thus, justifications that have more potential for supporting mathematics education are those with more explanatory power (Yackel & Hanna, 2003). This view is consistent with perspectives that 'an acceptable justification' is socially constituted through classroom practices (Bieda, 2011; Cobb, Stephan, McClain & Gravemeijer, 2001) and that students' reasoning is a tool for participating in classroom discourse (Greeno, 2003). Although empirical justifications may support students in making sense and solving mathematical problems, they have limited explanatory power (Knuth, Choppin & Bieda, 2009). Therefore, educators are encouraged to support students' development of more powerful justification schemes and flexibility to use different justification schemes to fit varying contexts.

Despite emphasis on justifications in different curricular, very few studies have attended to instructional practices in the classrooms (e.g., Reid & Zack, 2009). Of these few studies, the focus is mostly on teacher educators' practices (e.g., Komatsu, 2010). Even fewer are studies that looked at justifications in elementary schools. Thus, "the existing research knowledge base provides insufficient guidance about the ways in which reasoning and proving can be developed" (Stylianides & Silver, 2009; p. 249). Accordingly, exploring learning and teaching of justifications in elementary schools is still a fertile research ground. Moreover, Knuth, Choppin, Slaughter, and Sutherland (2002) explained that

engaging teachers in discussions focused on the details of students' competencies in justifying and proving may provide a basis for enhancing both teachers' own understandings of proof and their perspectives regarding proof in school mathematics. In addition, such detail on student reasoning may also provide a basis for continued growth and development of teachers' understandings of their students' reasoning and, consequently, their abilities to support the development of their students' mathematical reasoning (p. 1700).

Therefore, the goal of this study is to contribute to an understanding of justification in elementary schools by looking at what students do, what teachers do, and what should be encouraged. To serve this goal, this study explored justification schemes that elementary school students used when engaged in pattern finding activities. To this end, justification schemes were explored without regard to the frequency of each scheme since there is already ample evidence of students' reliance on empirical justifications. This delimitation was also decided upon after observing confounding factors in making claims about frequency of each scheme in different schools as these frequencies varied from task to task, and depended on teacher practice and student experiences. Another objective was to explore how elementary school teachers enact a curriculum that requires them to encourage students to justify their mathematical conjectures.

Finally, this study demonstrates the importance of moving beyond empirical schemes.

Conceptual Framework

Justification Schemes

Based on their empirical research and literature review, Harel and Sowder (1998, 2007) identified justification schemes or ways that students may use to ascertain for themselves or persuade others about the validity of mathematical ideas. Justifications are categorized as externally based, empirical, and analytic schemes. These categories are not mutually exclusive and must be understood with attention to the context in which students use them (Sowder & Harel, 1998).

Segal (2000) described externally based schemes as superficial views based on the form or the source of the argument. Students may cite reference materials or people they assume to be more knowledgeable than they are as basis for their justifications and not at the sense and correctness of the reasoning itself (i.e., authoritarian scheme). Students may also cite algorithmic procedures without evaluating the sense in those procedures in relation to the mathematical context (i.e., symbolic scheme). Another form of externally based scheme—ritual scheme— involves students using the form of arguments that have been socially established by the class as acceptable without evaluating the reasoning and content of the arguments. In general, students do not engage with the mathematics when they appeal to external authority (Simon & Blume, 1996).

\Unlike externally based schemes, students using empirical schemes show ownership of the justifications and mathematical engagement. "In an empirical proof scheme, conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (Harel & Sowder, 1998; p. 252). Students may use inductive reasoning to check the validity of assertions by

quantitatively evaluating a few cases or substituting a few numbers in algebraic expressions. Empirical justification schemes also include perceptual schemes, which are limited mental images or pictorial representations. These images are limited in that they do not attend to the task's general context and fail to anticipate transformations. An underlying feature of empirical justifications is lack of consideration or reference to the generality of the problem's context.

In contrast, analytic schemes use the general context of the problem. Analytic schemes use logical deductions to ascertain or persuade about the validity of conjectures (Sowder & Harel, 1998). Analytic schemes are divided into transformational and axiomatic schemes. Transformational schemes are goal oriented and anticipatory, involve images (including verbal or written statements) that show features of the general context, and use the transformations in the justification process (Harel & Sowder, 1998). These schemes make logical inferences and use operations in ways that anticipate changes. Axiomatic schemes use facts (assertions that have been justified), undefined terms (e.g., point), or statements accepted without proof, as the basis in the justification process.

Justification Curriculum

A mathematics curriculum is defined as a collection of mathematical tasks (Doyle, 1983). NCTM (2000) further explains that mathematical tasks in a curriculum are (or should be) coherent and promote in-depth understanding of connected mathematical ideas. A mathematical task, as described by Stein, Grover and Hennigsen (1996), is "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). As Stylianides (2008) described reasoning and proving activity as involving identifying patterns, making conjectures and providing justifications for those conjectures, a justification curriculum is operationally defined as a collection of tasks that

engage students in justifying conjectures from pattern-finding activities.

Stein et al. (1996) presented a framework that showed that written curriculum may change as it is implemented in the classroom. Factors contributing to these changes include teachers' choice of instructional materials, how the task is launched in the classroom, and students' implementation of the task. Drawing from this framework, Bieda (2010) studied enactment of proof related tasks in middle schools and analyzed proof tasks as written in textbooks, factors influencing how the teachers set up the tasks, and the associated learning outcomes. Similarly, Stein and colleagues focused on task setup, implementation and factors affecting implementation in grades 6 through 8 mathematics classrooms. This present study was conducted in elementary schools (grades 3 through 5) and focuses on pedagogical practices in enacting justification curriculum and justification schemes used by students. The following research questions guided the study:-

1. *How is a justification curriculum enacted in elementary schools?*
2. *What justification schemes do elementary school students use?*

It is necessary to study these two questions because justification is at the core of mathematical understanding making justifications a norm in mathematics classrooms is constrained by teacher experiences "defined largely by the memorization of facts and procedures" (Blanton & Kaput, 2008; p. 361). Understanding the different justification schemes students can use supports teachers' growth in content knowledge, understanding of students' reasoning, and development of their pedagogy (Knuth et al., 2002). Thus, there is a need to systematically explore the different ways in which students engage with justification tasks (Martinez, Brizuela & Superfinemake, 2011). In addition to understanding different justification schemes, understanding how a curriculum with emphasis on justifications may be enacted can inform teacher

educators in supporting teachers to develop routines of practice that develop justification sociomathematical norms (Cobb, Stephan, McClain & Gravemeijer, 2001).

Method

Context

 This study is part of the (name blocked out) project. This project aims at fostering elementary school students' mathematical reasoning through engagement in pattern finding activities in after school enrichment programs. Participating elementary school teachers attended 3 five-week blocks of professional development that focused on developing mathematical reasoning, content knowledge, knowledge of students' reasoning and productive pedagogical practices. Professional development participants met once a week for 90 minutes. During professional development meetings, teachers worked on the same mathematical tasks as they used in their after school classrooms, participated in and discussed the intended pedagogical practices. Teachers were given instructional materials to guide their enactment of the (project name blocked out) curriculum. This study reports practices of 3 teachers whose general teaching styles, based on the research team's observation before data analysis, seemed representative of the different styles of all (project name blocked out) teachers. Their elementary school teaching experience ranged from 6 to 22 years.

Data collection

Data for this study were collected after the teachers attended the first professional development session. Video and audio cameras recorded classroom activities of the teachers and the students. The cameras focused on the teachers and students during whole group and small group discussions. Each class had about 15 students. Students' written artifacts were collected at the end of each lesson. All instructional materials that were provided to the teachers were also collected and analyzed.

Mathematical problem

Students were expected to find the maximum number of people that can sit around a train of one, two, three, 10, and 100 train tables if one person sits on each side of the small table making up the train (see figures 1-3). They were asked to observe patterns, and to make and justify conjectures for number of people for n-table train.

Figure 1. Square Tables Train Task

Figure 2. Triangular Tables Train Task
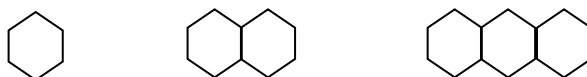
Figure 3. Hexagon Tables Train Task

Data analysis

Students' written and verbal transcripts were analyzed using Sowder and Harel's (1998) framework. Two raters independently coded the justifications. When there was a disagreement in the justification scheme codes, each coder gave a rationale for the codes by revisiting a description of the justification schemes until

an agreement was reached. While I agree with Sowder and Harel's position that all three types of justification schemes have an important place in mathematics education, the focus in this paper is delimited to empirical and analytic justification because it is only with these two justification schemes that students engage with the mathematics in attempts to justify their conjectures.

The framework on enactment of tasks guided the analysis of data on how the curriculum was enacted. Content analysis of the instructional materials was conducted to identify intended features of the (project name blocked out) curriculum. The content analysis identified major themes (e.g., taking up or creating opportunities to ask for justifications) and the corresponding subthemes (e.g., asking why and why not questions) that guided the rating of instructional quality. According to Matsumura, Garnier, Slater and Boston (2008), three raters are sufficient to rate quality of instruction. On a scale of 0 to 3 (0 for not enacted, 1 for barely enacted, 2 for almost adequately enacted and 3 for adequately enacted), three raters independently assessed the extent to which the teachers implemented the features of the intended curriculum. All raters attended professional development with teachers and participated in discussions of expected practices for enacting justification tasks in elementary classrooms. Inter-rater reliability was about .8 and above for all the teachers.

Qualitative methods were used to analyze practices of each case to identify themes. The videos and audio recordings of the classroom activities were transcribed. The data were entered into NVivo data software. After line-by-line coding that used active verbs to describe the classroom activities, the verbs were grouped into themes that summarized the classroom practices regarding justifications. NVivo was used to check if the themes that emerged robustly presented the practices of the teachers. Videos of the classroom activities were then used to check if the analysis of the transcripts were consistent with what was observed in the videos. The themes directed the narrative of each teacher's practices.

Results

The results of the analysis are divided into two sections organized by the two research questions that guided this study. The first section explores the different justifications used by elementary school students. The second section reports how three elementary school teachers enacted the tasks which students used to make and justify conjectures. These sections are followed by a discussion of the results.

Justification Schemes

As stated earlier, students were asked to predict how many people could sit around a train of 100 and any number of tables. For any number of tables, the general rules were equivalent to $2t + 2 = p$, $t + 2 = p$, and $4t + 2 = p$ where t represented number of tables making up the trains and p represented number of seats around each train for trains of square tables, triangle tables and hexagon tables respectively. Students used these rules to predict number of people that can sit around a train of 100 tables. In verifying their answers and justifying their rules, the following schemes were used. As stated earlier, these justification schemes should be understood in the context in which students used them and not as mutually exclusive.

Empirical schemes. Two empirical approaches were observed when students were justifying their general rules and when they verified their prediction of how many people can sit around a train of 100 tables. With the first approach, some students started their justification from the general rules (e.g., $2t + 2 = p$ for square table task). They used their rules to find outputs and compared the outputs from these rules with the outputs from counting number of people from the models of tables. Both actions—counting the number of people and trying out specific cases in the algebraic expression $2t + 2 = p$ are what Harel and Sowder (1998) refer to as quantitative evaluation which is a manifestation of inductive schemes. Episode 1 shows a student's work for verifying if the output values from the general rule are the same as those in the

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input-output table found by counting seats around a train model of square table tasks. After observing that their rules gave them correct output values for the first few cases, they concluded that their rules must be valid and the output values that they computed for larger input values using those rules must be true.

Episode 1

Square tables	People
1	4
2	6
3	8

Input-output table

$$(1 \times 2) + 2 = 4$$
$$(2 \times 2) + 2 = 6$$
$$(3 \times 2) + 2 = 8$$

Student reasoning

A second empirical approach involved justifying the general rules just as in the first approach. However, instead of using the rules to verify a prediction of number of students for a train of 100 tables, the students disregarded the explicit rules and wrote out all inputs and their corresponding outputs from 1 to 100 to confirm or disconfirm their prediction for 100 tables. Students using this scheme seemed to find the justifications that wrote out all the steps in between to be more convincing. Episode 2 contains some of a student's work for a train of 100 hexagon tables. This episode is an example of empirical justification because the students found it convincing based on the physical facts or sensory experiences.

Episode 2

98 tables	394 People
99 tables	398 Peopl
100 tables	402 People

Students were expected to justify parts of their rules. They were required to explain where the constant +2 in the general rules came from given context of the problems. Two approaches were observed, both of which focused on the action of building the train tables. Some focused on how many seats were being added each time a table was added to the train. For example, some students used their observation that in building a square tables train model, two seats were being added each time a table was added (see episode 3). This episode is classified as empirical because it fails to anticipate transformations. Consider the case in which shapes of the blocks building the trains were changed as an example. When students changed their tasks from trains of square tables to trains of triangle tables, the rules changed from $2t + 2 = p$ to $t + 2 = p$. Students using schemes like in episode 3 could not negotiate their rules to match the changes in the shapes and justify the +2 in $t + 2 = p$ when only one seat is added each time a triangle is added to the train.

Episode 3

The two comes from the 2 sides every time you add a table to a table

To justify $2t + 2 = p$ as a rule for a train of square tables, other students considered their perceptual observation that two seats were being removed from the sides at which the tables were being joined. Episode 4 from transcripts of students' discussion illustrates this reasoning. The mental image in episode 4 is classified as a perceptual scheme, a subcategory of empirical schemes, because students using this scheme were not able to connect the number of lost seats to the general context of the problem or the general rule ($2t + 2 = p$) that they were trying to justify. When other students asked those using this scheme how the number of lost seats related to the $2t + 2 = p$ rule, no explanation could be provided.

Episode 4

Brenda: Okay, this table (shows one table to a partner).

Dan: yes.

Brenda: Then you take away this (one side of the table) right here. They (two tables) are getting ready to be put together. So if you have this table and you put it together with this table you have to take this chair away (block one side of the first table) and on this table you have to take this chair away (block one side of the second table). And then you put them together.

Emerging analytic schemes. The defining features of analytic schemes include their attention to the generality of the contexts, the use of deductive and axiomatic schemes, and ability to anticipate transformations. The analytic schemes in this study are referred to as emerging because, as others may argue, these elementary school student schemes lack the rigor typically associated with deductive and axiomatic schemes. Attention to the features of the general context and shared characteristics with analytic schemes classifies these emerging analytic schemes beyond empirical schemes.

One of the emerging analytic schemes focused on the changing and the constant parts of the models of the train tasks. Students reasoned that there were always two seats on the end sides of the train regardless of the length of the trains (see episode 5), and the coefficient in the general rule corresponded to the number of seats that each table on the train was contributing. That is, the rule for the triangle tables train task is $t + 2 = p$ because each triangle table contributes one seat to the train and there is one extra seat at each end of the train. Students built different sized models to show the two end seats and to show the contribution of each table to the total number of seats for each train. Students were able to transfer this reasoning to other situations when the shape of the building blocks changed (see episode 6). This reasoning is classified as emerging analytic scheme because students reasoned

with the general features of the trains and the shapes making up the trains. Moreover, students were able to modify the reasoning when the shapes on the trains changed. Such reasoning is similar to what Harel and Sowder (2007) describe as anticipating transformations and using transformations in the justification process.

Episode 5
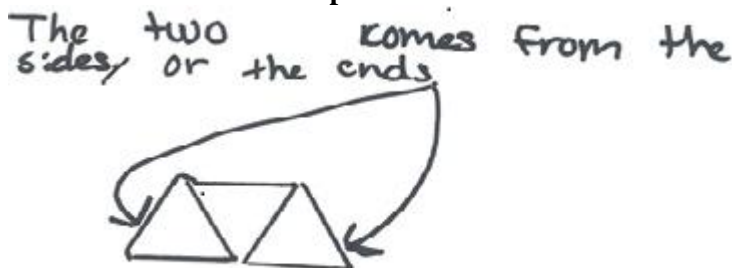
A slight variation to the focus on contribution of each table to the train was a focus on the top and bottom part of the train. Students considered contribution of each table to the top and then the bottom of the model. They used this reasoning to predict total number of seats at the top, at the bottom, and the ends of the train. This reasoning was used by a student in episode 6, in which a student was justifying his response to how many people can sit around a train of 100 hexagon tables. This student explained that because each table was contributing two seats at the top, then there are 200 seats available at the top of the train. Similarly, there are 200 seats available at the bottom. The image in episode 6 was used to explain this thinking. This reasoning was very common with square and hexagon trains but rare with the triangles. Like in episode 5, this reasoning used the general features of the train and the building blocks, and was applied when the train changed sizes and shapes.

Episode 6

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Another emerging analytic approach transferred the reasoning from square tables task to other tasks. In episode 7, a student was justifying that *4t + 2 = p* is a rule for hexagon tables and 402 seats are available on a train of 100 hexagon tables. This student considered the general characteristics of the square tables task and how it was previously justified. He transferred that reasoning and modified it to fit his present context to argue for number of people that can sit around a train of 100 hexagon tables. Student reasoning in episode 7 has been rearticulated for clarity of communication because as Reid (2002) noted, young students have difficulties articulating their deductive reasoning. Nevertheless, the student is using deductive reasoning, a form of analytic schemes.

Episode 7

Student written justification	+x4+2=p Because a square has 100 on each side and it has one side so a hexagon has 2 sides so it has 200 on each side.
Rearticulated justification	1. On a train of square tables, each square contributes one seat on each side, and the train has two seats on the ends. 2. The rule for a train of square tables is *t + t + 2 = p or 2t + 2 = p*. 3. On a train of hexagon tables, each hexagon contributes two seats on each side, and the train has two seats on the ends. 4. Therefore, the rule for a train of hexagon tables must be *2t + 2t + 2 = p or 4t + 2 = p*. 5. Following the same reasoning, a train of 100

| | square tables has 100 seats at the top and 100 seats at the bottom, and two seats on the ends. 6. A train of 100 hexagon tables then has 200 seats at the top, 200 at the bottom, and two seats on the ends. |

A different emerging analytic justification was an attempt of justification by contradiction in episode 8. It must be noted that the equal sign was incorrectly used in this episode. When predicting number of people that can sit around a train of 100 square tables, some predicted 220 and others 202. In justifying that 202 was the correct number of people who could sit around 100 tables, the student in Episode 8 applied an assertion that he justified using a model similar to the one discussed in episode 6 and the rule $2t + 2 = p$ which was already justified as valid for square table trains task. Such reasoning is similar to what Harel and Sowder (2007) described as axiomatic reasoning.

Episode 8

$$100+100=200+2=202 \text{ not } 220$$
$$100\times2= 200+2=202 \text{ not } 220$$
$$+2 \text{ not } 20$$
$$cannot=220 \neq 200+2$$
$$100+100+2$$
$$to=220 \text{ is } 100+100+20$$

These results show that elementary school students are capable of using different schemes to justify their conjectures including emerging powerful schemes that could be nurtured and used to refine and develop their justification fluency. The following sections report how justification curriculum was enacted in elementary classrooms. Content analysis of (project name blocked out) instructional materials showed several features that aimed at

developing and nurturing students' justification schemes. This paper focuses on enactment of features categorized as creating or taking up opportunities to (1) ask for justifications and (2) develop different justification schemes. Results from cross-case analysis are followed by results from analysis of each case.

Cross-case Analysis of Enactment of Justification Curriculum Creating or Taking up Opportunities to Ask for Justifications

Intended features. Three features were identified in this category. Teachers were expected to ask students for convincing arguments about validity of their rules. In addition, they were expected to ask students to evaluate peers' responses and question each other. Teachers were also expected to use student's conflicting responses as opportunities for students to justify their rules to each other.

Enacted features. Three raters observed the lesson and assessed the extent to which each of the teachers enacted these intended features. Average ratings of the extent to which each teacher enacted task features are presented in Table 1. From the ratings, while one teacher adequately asked students what they thought about their peers' ideas and strategies, the other teachers did not. Using conflicting responses as an opportunity for students to justify to each other was the least enacted feature.

Table 1. Average Ratings on How Teachers Took up or Created Opportunities to Ask for Justifications

The extent to which teachers took up or created opportunities to :-	Teacher		
	1	**2**	**3**
Ask why and why not questions	2	1	1
Ask students what they think about peers' answers	3	.6	.6
Use conflicting responses as opportunities for students to justify to each other	1	0	0
Ask students to question each other	1.33	.33	1

Creating or Taking up Opportunities to Develop Different Justification Schemes

Intended features. (project name blocked out) instructional materials encouraged teachers to support students' different ways of justifications. Teachers were encouraged to support students' analytic justifications that referred to the general context of the mathematical problem by asking students to explain how parts of their rules connected to the contexts of the tasks. Teachers were discouraged from being judges of the validity of responses. Rather, teachers were supposed to ask students to convince each other of the validity of their ideas. Thus, authoritarian schemes were discouraged.

Enacted features. As intended, teachers' (2/3) instructional habits did not at all encourage students to refer to others as sources of validity (see Table 2). Empirical justifications were the most enacted schemes. On the other hand, analytic justifications were barely enacted.

Table 2. Average Ratings on How Teachers Took up or Created Opportunities for Students to Develop Different Justification Schemes

The extent to which teachers took up or created opportunities for students to:-	**Teacher**		
	1	**2**	**3**
Develop analytic justifications by connecting the rule to the model (e.g., ask where the +2 in 2n+2 comes from)	1	1.33	0
Develop empirical justifications (e.g., nurture example based justifications or actual counting of seats using manipulatives or drawings)	2.66	1	2
Develop authoritarian justification schemes (e.g., nurture 'it is true because my teacher said')	0	1	0

Enactment by Each Case

Teacher 1. Teacher 1 asked "why" questions and her students expected their peers to justify their conjectures. For example, she asked a student to "prove (that 202 people would sit around 100 square tables) without drawing 100 tables." When this student hesitated, the other students chanted, "Prove it! Prove it! Prove it!" Justifying by counting the actual seats was discouraged but example based justifications and justifications by drawing the general context of the problem were privileged. After 2 students justified their rules in different ways, Teacher 1 said, "So, it looks like his (student's) rule works. And he proved it. He proved it with kind of a picture in mind, right? (She) proved it (her rule) with a formula in mind (by trying out 3 examples)." However, students tended to use a few examples as a sufficient justification and in such cases the teacher did not push students to justify if their rule worked for *any* number of tables. Teacher 1 rarely asked students to question each other, but she often asked students to evaluate peers' rules by applying them to check if their strategies yielded similar results.

Teacher 2. Teacher 2 seldom asked students to justify their conjectures. She sometimes said to the students: "make sure you explain. There it says convince me your rule works. Don't just say I know it works." In this classroom, asking for a justification had a form of "explain how you got your answer." Teacher 2 did not directly ask students to question each other, but encouraged students to explain their conjectures to peers in ways that peers would easily understand. Additionally, after the task was launched, students worked the problem out and wrote down their responses, but whole class discussions did not follow. This lesson design did not take up any opportunities to use conflicting responses for justification.

Teacher 3. Teacher 3 asked for justifications in both small and whole group discussions. In episode 10, students were justifying conjectures on how many people would sit around 100 and *n*

tables. When students shared conflicting responses, Teacher 3 praised student thinking but did not take up the opportunity for students to justify to each other (see episode 10). She did not point out incorrect answers and she did not create opportunities for students to find out why their responses were not correct. While ignoring the incorrectness of some responses, Teacher 3 created a context for celebrating correct responses, especially when different student strategies arrived at the same answer. Students with correct responses were asked if other students' (correct) responses were valid. That is, only the students with correct answers were publicly given opportunities to evaluate other correct responses. In general, the class seemed content with example-based justifications.

Episode 10
Student 1: I just multiplied 100 by 2.
Teacher 3: So why did you multiply 100 by 2?
Student 1: Because my rule was multiply by 2.
Teacher 3: You are thinking (calls student 2 to present his rule).
Student 2: My rule is x 2 +2 (writes on the board) because 1 x 2 equals 2, plus 2 equals 4.
Teacher 3: Ok try the next one (input).
Student 2: 2 x 2 equals 4, plus 2 equals 6 and then 3 times 2 equals 6, plus 2 equals 8.
Teacher 3: Ok. So how about somebody who said 100 tables equals 202 people. Was that right then?

Discussion and Conclusions

This study explored justification schemes that elementary school students used when working on pattern finding tasks. The results of the analysis show students' different empirical and emerging analytic schemes. Students may also use externally based schemes that neither show ownership of the justifications nor engagement with the mathematics involved. This study has shown that, despite

students' demonstrated ability, most classroom practices do not nurture students' use of mathematically powerful justification schemes. Understanding different student justification schemes can support informed teaching practices that focus on student thinking to develop justification fluency. This section discusses the results and the implications for research and practice.

Attempts to develop students justification fluency can draw from students' natural tendencies to ask why questions. This may require teachers to create inquiry classroom contexts in which students see that they construct their own understanding. Such contexts are atypical as revealed in Good's (2010) literature review from 1968 through 2008 that normative practices of classroom teachers are teacher centered and teachers usually position themselves as sources of knowledge. Such normative practices promote authoritarian or externally based justification schemes. One way of developing contexts that support justifications is by having small group discussions in which students take the responsibility of understanding strategies by peers, and of making peers understand their reasoning (Store, 2014). Other discursive practices that produce shared learning authority in the classroom may be productive as well.

The types of questions by teachers in this study were highly associated with the variety of schemes used. For example, justification by contradiction was only evident when students had to decide the correctness of one answer against the other. Students also tended to move from empirical schemes to emerging analytic schemes when asked to explain parts of their rule. Thus, if the goals of supporting sense making through justifications are to be realized, teachers must develop their art of questioning. Teacher educators should focus on supporting teachers and future teachers to examine the types of questions they use and those they need to use for different scenarios in the classroom. For example, a habit of asking good justification questions may develop if teachers write down such questions in their lesson plans. Teachers may then reflect on the extent to which their questioning provoked

convincing arguments; students' constructed meanings of what convincing arguments are; and their flexibility in turning different classroom scenarios into justification opportunities.

Another important practice in developing students' justification fluency is connecting student reasoning to the context of mathematical tasks (Store & Store, 2013). Participating teachers in Stylianides's (2008) study reported that when students focused on the models or context of the tasks when making conjectures from pattern-finding tasks, they tended to use analytic justifications. In contrast, the teachers that guided students to use input-output tables to make mathematical conjectures saw more empirical justification schemes. In the current study, the general perceptual justification scheme was commonly used with square and hexagon train tasks, but not with triangle trains. This is because triangles did not present an obvious geometric model that showed the number of seats in relation to student rules. These two studies show the relationship between types of justification schemes and the focus on the context of the problem. Teachers may support students' contextual reasoning through the choice and sequence of tasks, and through instructional practices reported in Stlyianides's study.

This study has explored a topic—teaching and learning justifications in elementary schools—that is still emerging. Jones (2000) argued that the "key issue for mathematics education is how children can be supported in shifting from 'because it looks right' or 'because it works in these cases' to convincing arguments which work in general" (p. 55). This is necessary especially because classrooms like in this study overly rely on empirical justifications. At the same time, I argue that empirical justifications, including those that involve trying a few cases, should be accepted and used as tools for promoting sense making and mathematical reasoning in elementary classrooms. Stylianides and Silver (2009) wrote that enacting justification tasks is both a pedagogical and content problem for teachers. In the current study, the teachers were able to use different justification schemes and

appreciated the value of analytic schemes, but their classrooms relied on empirical justifications. This adds another dimension to the complexity of preparing teachers to enact curriculums that require justification of ideas. This calls for further research that inform transferring of theoretical knowledge to teaching practice, and developing teaching habits that nurture development of justification fluency. Focusing student attention to the context of the tasks, fostering productive small group discussions, developing teachers' reflective habits about their questions, and understanding justification strategies that students may use for different tasks are just a few examples that may improve practice.

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